UNCLASSIFIED

AD NUMBER
AD005767
CLASSIFICATION CHANGES
TO: **unclassified**
FROM: **restricted**
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to DoD only; Administrative/Operational Use; DEC 1952. Other requests shall be referred to Office of Naval Research, Arlington, VA 22203. Pre-dates formal DoD distribution statements. Treat as DoD only.**
AUTHORITY
E.O. 10501 dtd 5 Nov 1953; ONR ltr dtd 26 Oct 1977

THIS PAGE IS UNCLASSIFIED

Code 421

RESTRICTED

REPORT NO. M720-1 R 37
RESTRICTED
COPY NO. 2

ENGINEERING RESEARCH INSTITUTE

LAC

THE SUBMARINE APPROACH PROBLEM

BY

EARL CRISLER

MELBOURNE STEWART

DIRECTOR OF PROJECT

A. H. COPELAND, SR.
PROFESSOR OF MATHEMATICS

CONTRACT N6 ONR 232-1

PROJECT M720-1

DECEMBER, 1952

SECURITY INFORMATION

RESTRICTED

UNIVERSITY OF MICHIGAN • ANN ARBOR

RESTRICTED

REPORT NO. M720-1 R 37
RESTRICTED
COPY NO. 2

THE SUBMARINE APPROACH PROBLEM

BY

EARL CRISLER

MELBOURNE STEWART

DIRECTOR OF PROJECT

A. H. COPELAND, SR.
PROFESSOR OF MATHEMATICS

CONTRACT N6 ONR 232-1

PROJECT M720-1

DECEMBER, 1952

SECURITY INFORMATION

RESTRICTED

DISTRIBUTION LIST

Office of Naval Research
Navy Department
Air Branch, Code 461
T/3 Building, Room 2609
Washington 25, D. C. ....Copies 1 - 10

Office of Naval Research
New York Branch Office
346 Broadway
New York 10, New York ....Copy 11

Director, Office of Naval Research
Chicago Branch Office
844 N. Rush Street
Chicago 11, Illinois ....Copy 12

Director, Office of Naval Research
San Francisco Branch Office
1000 Geary Street
San Francisco 9, California ....Copy 13

Commanding Officer, Office of Naval Research
New York Branch Office
Building 3, 10th floor
New York Naval Shipyard
Brooklyn 1, New York ....Copy 14

Operations Evaluation Group (Op 374)
Office of the Chief of Naval Operations
Navy Department
Washington 25, D. C. .... Copies 15 - 16

Director, Naval Research Laboratory
Naval Gun Factory
Washington 25, D. C. .... Copy 17

Commander, Operational Development Force
USS Adirondack, U. S. Naval Base
Norfolk, Virginia ....Copy 18

Chief of Staff, U. S. Army
DC/S P & R
3-D-640 Pentagon
Washington 25, D. C. ....Copy 19

Mr. R. A. Bailey
Military Operations Research Division
Lockheed Aircraft Corporation
Burbank, California ....Copy 20

Chief, Bureau of Aeronautics
Navy Department
Washington 25, D C.
Att: Mr. B. A. Wiener ....Copy 21

Chief, Bureau of Ordnance
Navy Department
Washington, D. C.
Att: Re 9 ....Copy 22

Chief, Bureau of Ordnance
Navy Department
Washington, D. C.
Att: Re 2 ....Copy 23

Chief of Staff, U. S. Air Force
Washington, D. C. ....Copy 24

Chief of Staff, U. S. Army
Office of Chief of Ordnance
Washington, D. C. ....Copy 25

Cornell Aeronautical Laboratory, Inc.
4455 Genessee Street
Buffalo 5, New York
Att: Mr. R. Shatz ....Copy 26

RAND Corporation
1500 Fourth Street
Santa Monica, California
Att: Mr. Frank R. Collbohm ....Copy 27

Engineering Research Institute
University of Michigan
Ann Arbor, Michigan ....Copy 28

Armed Services Technical Information Agency
Document Service Center
U. B. Building
Dayton 2, Ohio
Att: DSC - SA ....Copies 29 - 33

Dean Walter Bartkey
Air Weapons Research Center
University of Chicago
Chicago, Illinois ....Copy 34

Guided Missiles Library, 22-001
Massachusetts Institute of Technology
Cambridge 39, Massachusetts ....Copy 35 - 36

Dr. E. D. McAlister
Eastman Kodak Company
Navy Ordnance Division
121 Lincoln Avenue
Rochester 11, New York ....Copy 37

Dr. W. L. Barrow, Chief Engineer
Sperry Gyroscope Company, Inc.
Grat Neck, New York ....Copy 38

Professor G. P. Wadsworth, Room 2-285
Massachusetts Institute of Technology
Cambridge , Massachusetts ....Copy 39

Commanding Officer, 3151st Electronics Station
Watson Laboratories, AMC
Red Bank, New Jersey
Att: WLEPL - 1 ....Copy 40

Contract N6 ONR 232-1
Project M720-1
University of Michigan
Ann Arbor, Michigan .... Copies 41 - 50

THE SUBMARINE APPROACH PROBLEM

1. Problem and Results

The Problem:

A submarine X sights a vessel Y. X determines Y's course, speed, and position. He then must decide whether it is possible to intercept Y or not. If X can intercept Y, then he is interested in knowing the courses and speeds that are available to him for making the interception. In particular, he would like to know the course and speed that he should use so that the energy remaining in his batteries after the interception is the greatest possible.

Figure 1 illustrates the generic situation with which X is confronted when an interception is possible. Here V is Y's speed, U is the speed X must travel for time T to intercept Y at P along the course β, D is the original distance between X and Y, and α is the angle X makes with Y's bow.

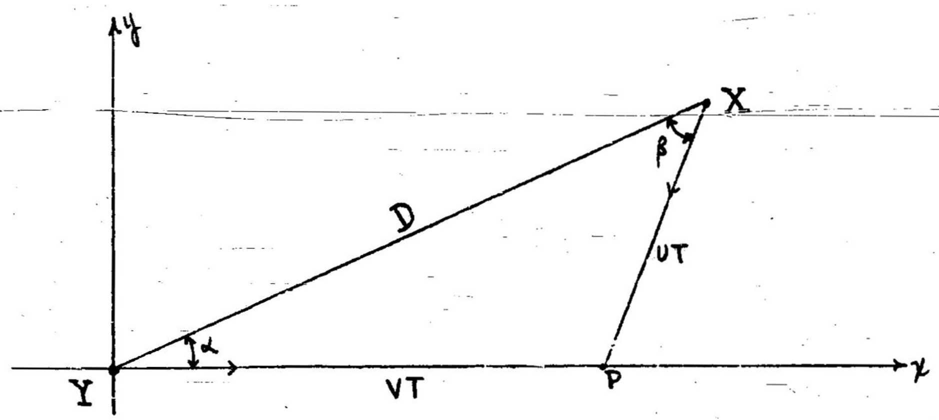


FIGURE 1

The following assumptions and restrictions are imposed on X and Y:

a) The energy available to X from his batteries in a time T (T given in hours) is :

$$(1.1) \qquad E_A(T) = E_\infty(1 - e^{-\sqrt{T/b}}).$$

E_∞ is the total energy in X's batteries and can range between 5,000 and 15,000 kilowatt hours. b is a constant which depends upon the battery construction and can range between 2.5 and 4.5 hours.

b) The energy that X will expend while traveling at the speed U for one hour is:

$$(1.2) \qquad P = cU^{\sigma}.$$

Consequently the energy expended by X in traveling a time T is given by

$$(1.2^*) \qquad PT = cTU^{\sigma}.$$

P is in kilowatts. c is in kw/(knots)$^\sigma$ and ranges between 0.5 and 2.5.

c) It is further assumed that Y's track is a straight line and X cannot travel at a speed in excess of 18 knots.

The Results:

In Section 2, with the preceding assumptions the question as to whether or not X can successfully intercept Y is answered. This is accomplished by constructing a contour about Y so that if X is inside of or on the contour, then he is able to intercept Y. If X is outside of the contour, an interception is impossible. Three methods for determining this contour are discussed.

Section 3 discusses a maximality criterion and derives some relations which are used in the later sections.

In Section 4, a relation of Section 3 is employed to determine the "minimal energy" contour, a curve which lies in the interior of the interception domain. It possesses the property that if X lies inside of or on it, then there exists a unique course $\overline{\beta}$ which depends only on the angle α and such that the energy expended by X in making an interception with the course $\overline{\beta}$ is less than the energy expended by X in making an interception with any other course. Clearly it is to X's advantage to use $\overline{\beta}$ for his interception course whenever this is possible.

Section 5 discusses approximations to the interception contour.

The report concludes with Section 6 which considers the problem of calculating the probability of X being able to successfully intercept Y.

An appendix lists the partial derivatives of the equations of the interception contour and minimal energy contour with respect to the parameters of the problem.

2. The Interception Contour

Let T be some fixed time. Formula (1.1) tells us that X has available in this period of time the energy $E_A(T)$. Upon combining (1.1) and (1.2*), we obtain the greatest speed at which X can travel for the period of time T, namely:

$$(2.1) \qquad \overline{U} = \left[\frac{E_A(T)}{cT}\right]^{1/6}$$

The symbol $\overline{U}$ shall mean throughout the remainder of this report that X's speed is given by (2.1).

We note that when $\bar{U}$ is used as X's speed in an interception of duration T, this is equivalent to assuming that the energy used by X is equal to the energy available to X. For (2.1) may also be written in the form

$$c T \bar{U}^{\sigma} = E_A(T).$$

Clearly $\bar{U}T$ is the greatest distance X can travel in the time T. Hence if we draw a circle of radius $\bar{U}T$ and center P as illustrated in Figure 2, we see that X can intercept Y from every point within and on the circle. On the other hand, if the period of time for the interception is to be T, then it is impossible for X to intercept Y from any point that lies outside of this circle.

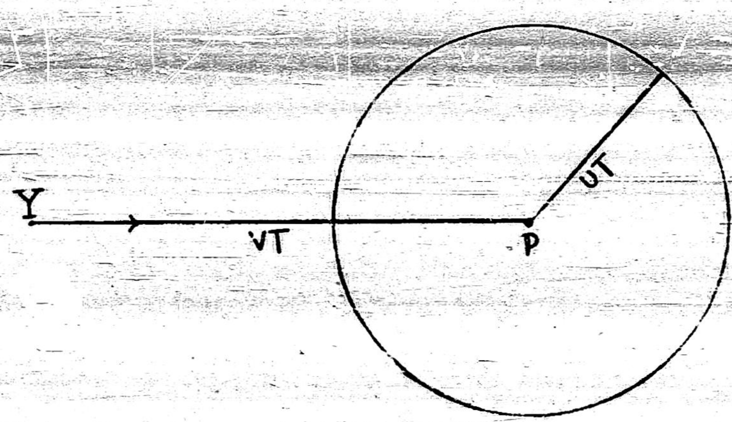


FIGURE 2

For every period of time T we obtain a configuration similar to Figure 2. Now if X can intercept Y, then he must be able to do it in some definite period of time. Thus if we let T vary over all positive values and impose the resulting configurations upon one another, we obtain a region which consists of all the points from which X can intercept Y. This region may be approximated by taking only a finite number of times, superimposing the resulting configurations upon each other, and constructing the convex hull of the resulting domain.

Figures 3, 4, 5, and 6 on the following pages are illustrations of this method of approximating the contour. Table 1 which precedes them gives the calculations of the $\overline{U}T$'s which were used in their construction. Due to the restrictions imposed by the size of the paper on which this report is reproduced, the interception contours have been closed off for 30, 30, 24, and 20 hours respectively. In Figure 3 where Y's speed was taken as 5 knots, the contour is a circle because of the time restriction. Since

$$\overline{U}T = (E_\infty/c)^{1/\sigma} \, T^{\frac{\sigma-1}{\sigma}} \left[1 - e^{-\sqrt{T/b}}\right]^{1/\sigma} \tag{2.2}$$

is a monotone increasing function of T, it is fairly apparent that the y-coordinate of the contour increases indefinitely with the time.

From the previous discussion it is seen that the interception contour is the envelope of a family of circles. A generic member of this family is given by the equation

$$(x - VT)^2 + y^2 - (\overline{U}T)^2 = 0, \tag{2.3}$$

where the origin for the frame of reference is Y's original position.

The equations of the envelope to the family are found by differentiating (2.3) with respect to time and taking the resulting equation together with (2.3), i.e.,

$$\begin{cases} (x - VT)^2 + y^2 - (\overline{U}T)^2 = 0 \\ V(x - VT) + \overline{U}T \dfrac{d}{dT} \left[\overline{U}T\right] = 0 \end{cases} \tag{2.4}$$

Thus the interception contour is given by the equations

$$\begin{cases} x = VT - \dfrac{\overline{U}T}{V} \dfrac{d}{dT} \left[\overline{U}T\right] \\ y = \pm \overline{U}T \sqrt{1 - \left(\dfrac{1}{V} \dfrac{d}{dT}\left[\overline{U}T\right]\right)^2} \end{cases} \tag{2.5}$$

where $\overline{U}T$ is given by (2.2)

T	T/b	$\sqrt{T/b}$	$e^{-\sqrt{T/b}}$	$1-e^{-\sqrt{T/b}}$	$E_A(T)$	$\frac{E_A(T)}{cT}$	U	UT
.20	.0800	.2828	.7537	.2463	2463.	12320.	23.10	4.620
.40	.1600	.4000	.6703	.3297	3297.	8242.	20.20	8.080
.60	.2400	.4899	.6127	.3873	3873.	6455.	18.62	11.17
.80	.3200	.5657	.5680	.4320	4320	5400.	17.54	14.03
1.00	.4000	.6324	.5313	.4687	4687.	4687.	16.74	16.74
1.50	.6000	.7746	.4609	.5391	5391.	3594.	15.32	22.98
2.00	.8000	.8944	.4088	.5912	5912.	2956.	14.35	28.70
3.00	1.200	1.095	.3344	.6656	6656.	2219.	13.04	39.12
4.00	1.600	1.265	.2822	.7178	7178.	1794.	12.15	48.60
5.00	2.000	1.414	.2432	.7568	7568.	1514.	11.48	57.40
6.00	2.400	1.549	.2125	.7875	7875.	1312.	10.95	65.70
7.00	2.800	1.673	.1877	.8123	8123.	1160.	10.51	73.57
8.00	3.200	1.789	.1671	.8329	8329.	1041.	10.13	81.04
9.00	3.600	1.897	.1500	.8500	8500.	944.4	9.811	88.30
10.00	4.000	2.000	.1353	.8647	8647.	864.7	9.527	95.27
11.00	4.400	2.098	.1227	.8773	8773.	797.5	9.273	102.0
12.00	4.800	2.191	.1118	.8882	8882.	740.2	9.046	108.6
13.00	5.200	2.280	.1023	.8977	8977.	690.5	8.839	114.9
14.00	5.600	2.366	.0939	.9061	9061.	647.2	8.650	121.1
15.00	6.000	2.449	.0864	.9136	9136.	609.1	8.477	127.2
16.00	6.400	2.530	.0797	.9202	9202.	575.1	8.316	133.0
17.00	6.800	2.608	.0740	.9260	9260.	544.7	8.167	138.8
18.00	7.200	2.683	.0683	.9317	9317.	517.6	8.029	144.5
19.00	7.600	2.757	.0634	.9366	9366.	492.9	7.899	150.1
20.00	8.000	2.828	.0591	.9409	9409.	470.4	7.772	155.4
21.00	8.400	2.898	.0551	.9449	9449.	450.0	7.663	160.9
22.00	8.800	2.966	.0515	.9485	9485.	431.1	7.555	166.2
23.00	9.200	3.033	.0482	.9518	9518.	413.8	7.452	171.4
24.00	9.600	3.098	.0451	.9549	9549.	397.9	7.355	176.5
25.00	10.00	3.162	.0423	.9577	9577.	383.1	7.263	181.1
26.00	10.40	3.225	.0398	.9602	9602.	369.3	7.174	186.5
27.00	10.80	3.286	.0374	.9625	9625.	356.5	7.091	191.4
28.00	11.20	3.347	.0352	.9647	9647.	344.5	7.010	196.3
29.00	11.60	3.406	.0332	.9667	9667.	333.3	6.933	201.0
30.00	12.00	3.464	.0313	.9687	9687.	322.9	6.860	205.8

$b = 2.5$ $E_\infty = 10^4$

$c = 1$

$\sigma = 3$

TABLE 1

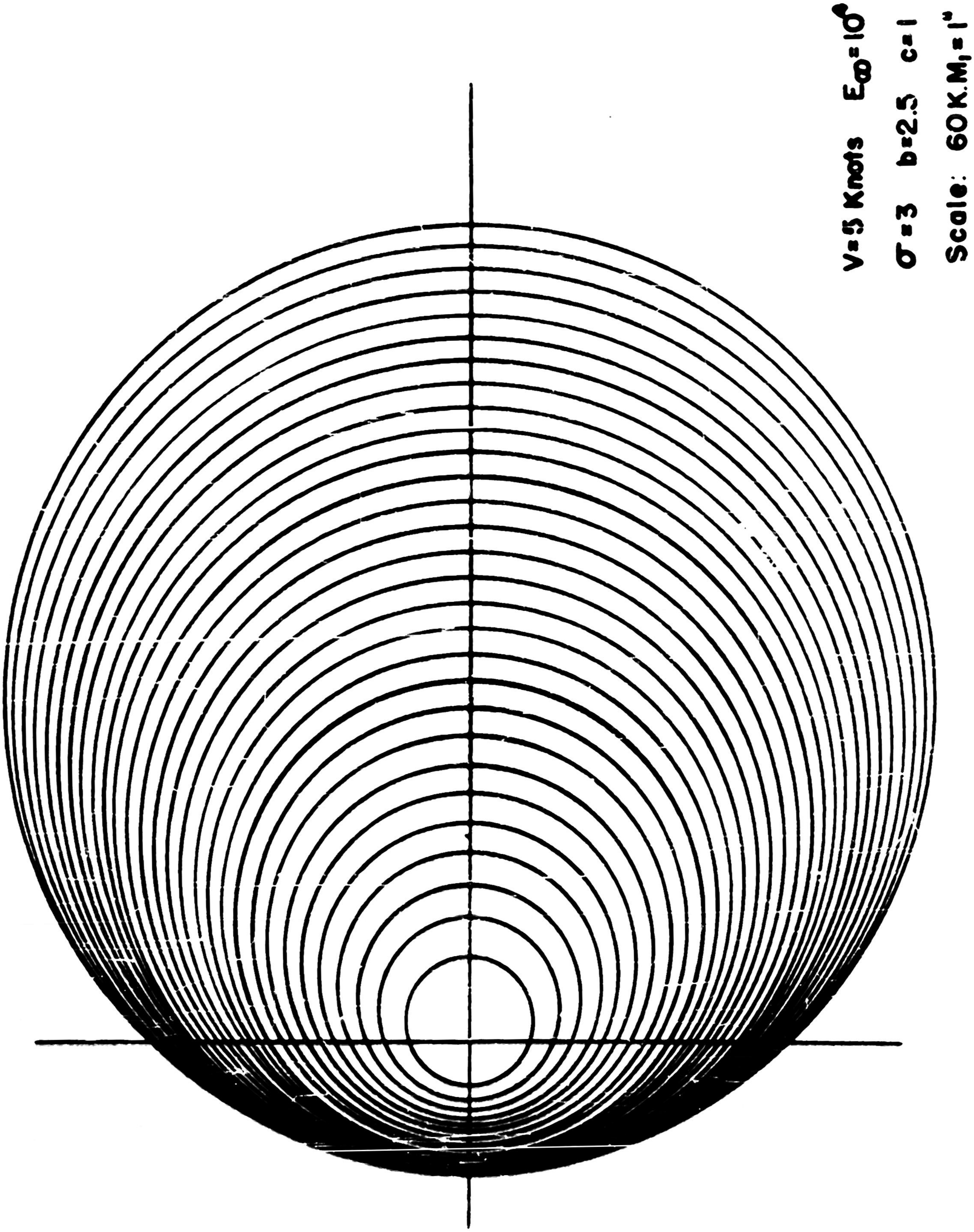


FIG. 3

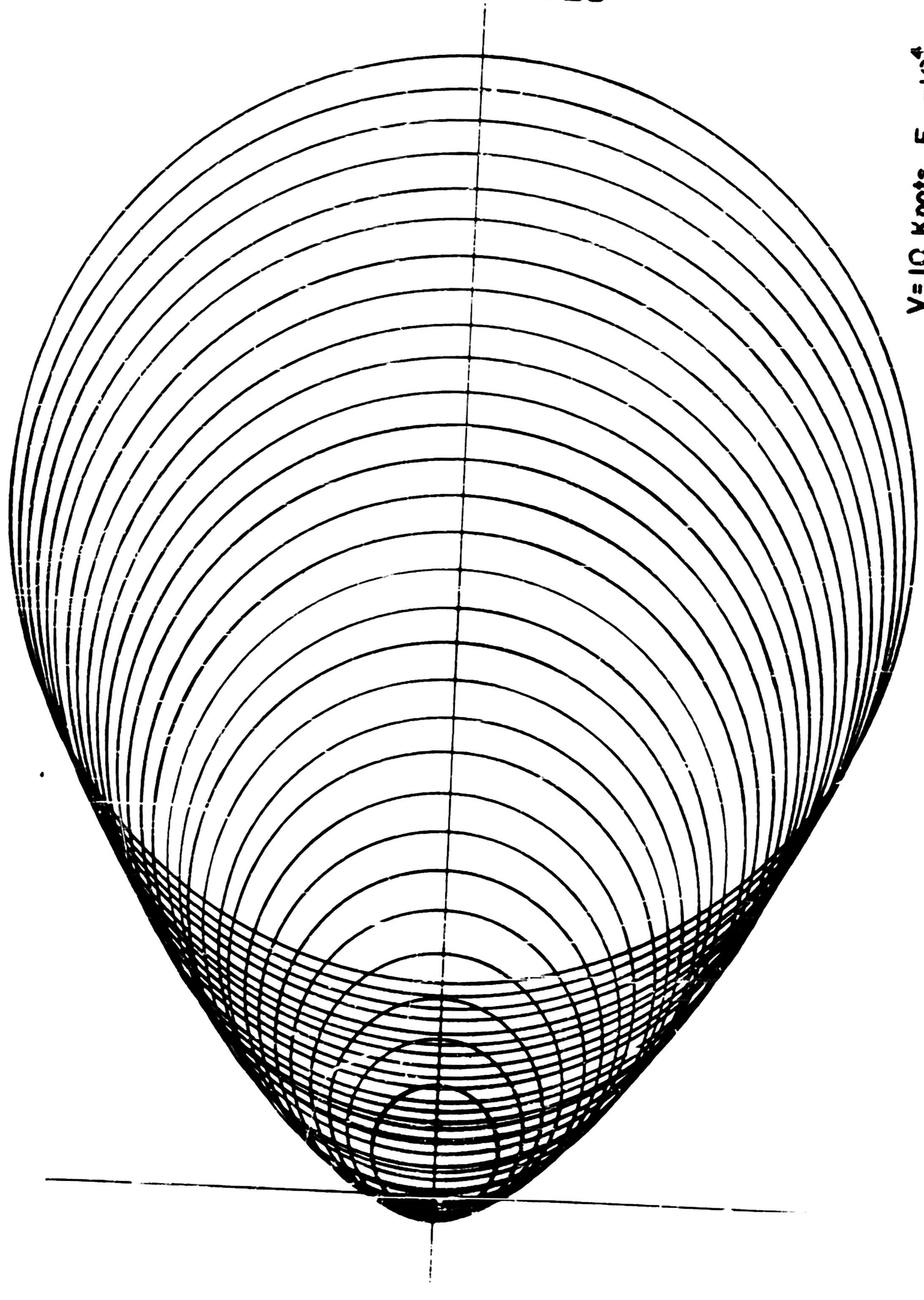


FIG. 4

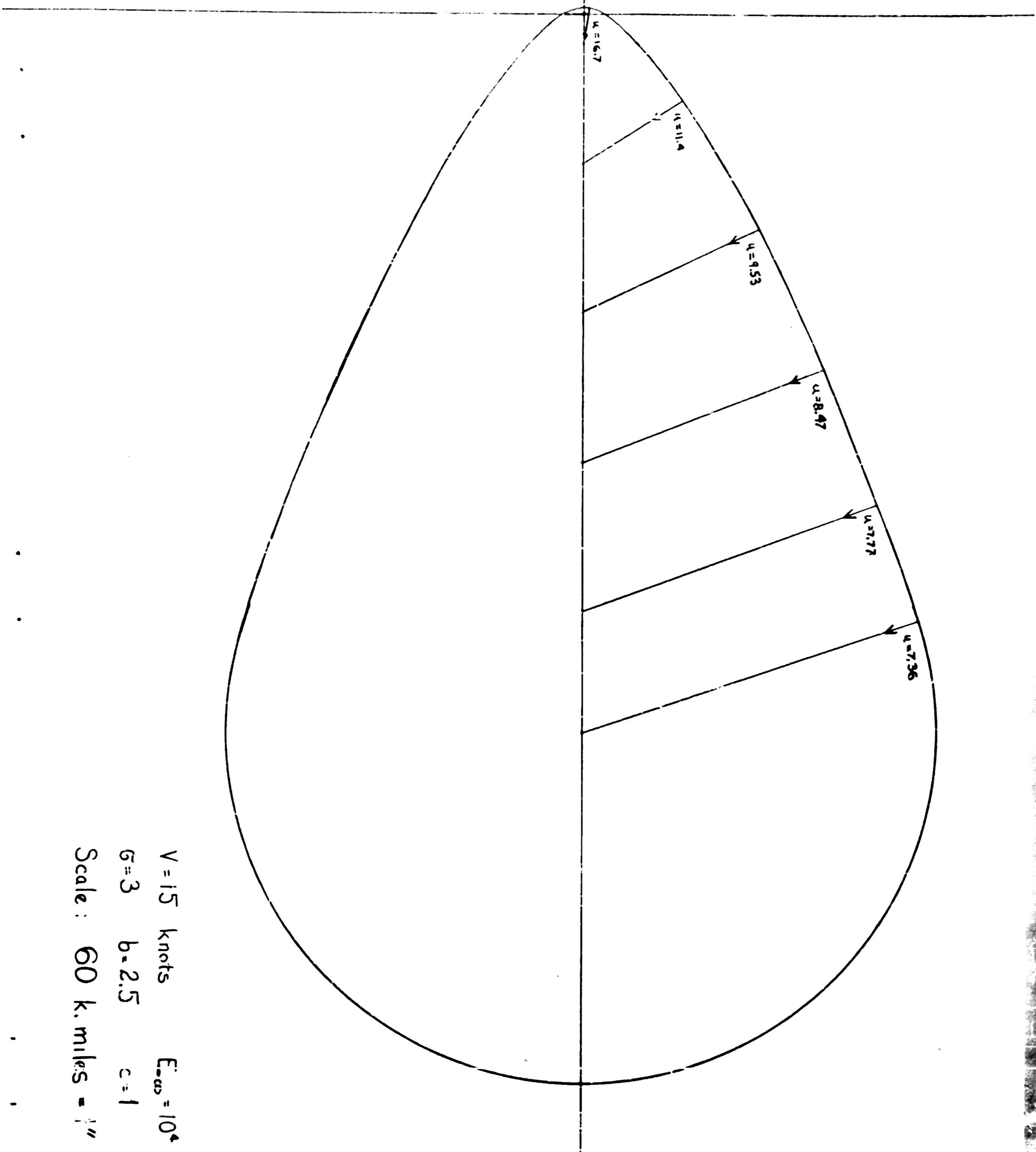
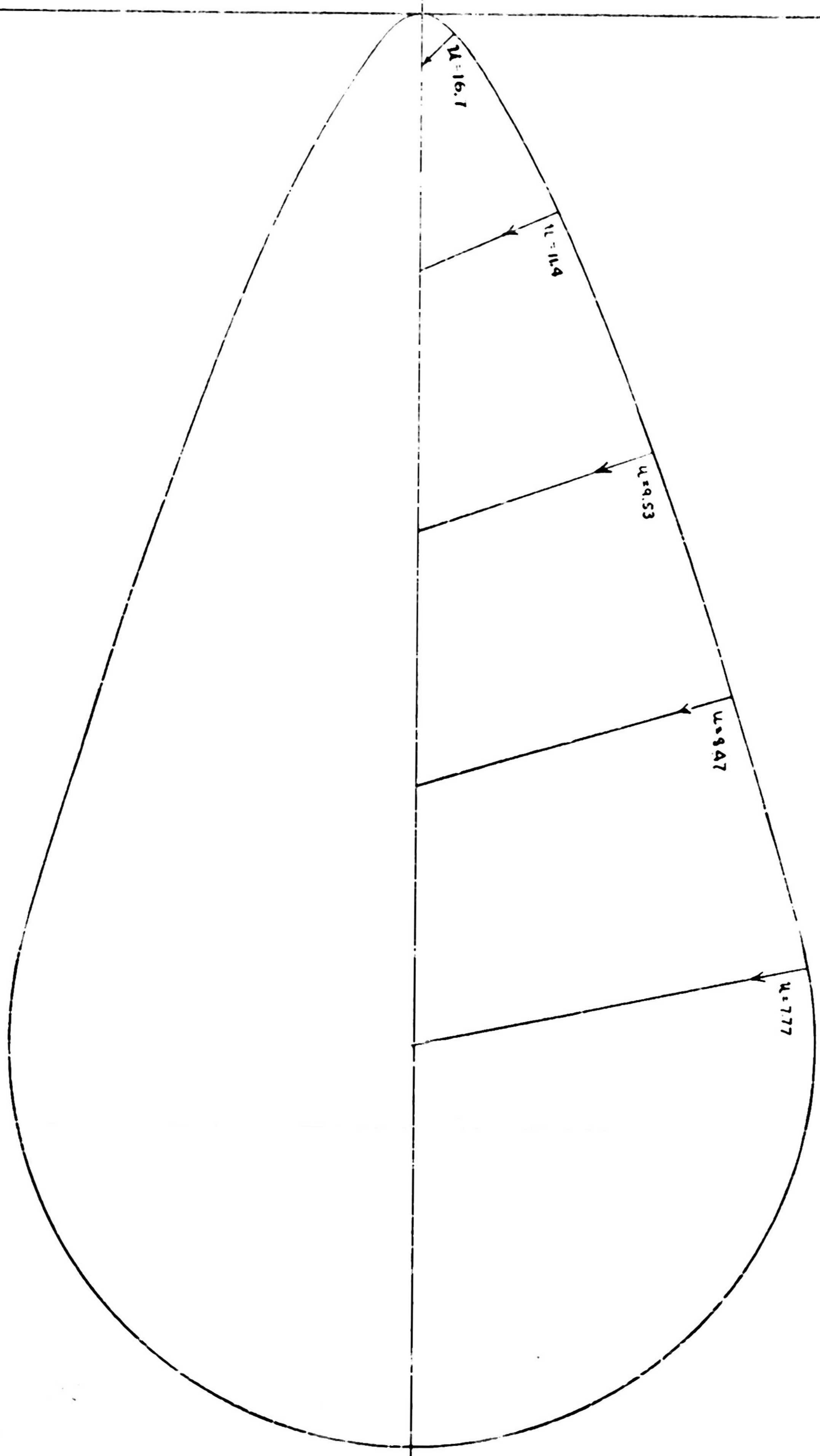

u = 16.7
u = 11.4
u = 9.53
u = 8.47
u = 7.77
u = 7.36
V = 15 knots $E_{-\infty} = 10^4$
σ = 3 b = 2.5 c = 1
Scale: 60 k. miles = 1"

FIG. 5



V = 20 knots $E_{\alpha} = 10^4$

G = 3 b = 2.5 c = 1

Scale: 60 k. miles = 1"

FIG 6

Hence another method of determining the interception contour would be to plot the function $\overline{UT}$ against T and from this curve find $\frac{d}{dT}\left[\overline{UT}\right]$ graphically. These values may then be used in (2.5) to calculate the contour. Or since (2.5) may be written in the form

$$(2.6)\qquad \begin{cases} x = VT - \frac{1}{2V}\frac{d}{dT}\left[\overline{UT}\right]^2 \\ y = \pm\sqrt{\left[\overline{UT}\right]^2 - \left(\frac{1}{2V}\frac{d}{dT}\left[\overline{UT}\right]^2\right)^2} \end{cases}$$

$\left[\overline{UT}\right]^2$ may be plotted against T and the same procedure used. We have not tried this method for calculating the contour and thus do not know whether it has any merit or not.

If formula (2.2) is differentiated with respect to time the result may be put in the form

$$(2.7)\qquad \frac{d}{dT}\left[\overline{UT}\right] = \frac{U}{\sigma}\left[(\sigma - 1) + \tfrac{1}{2}\sqrt{T/b}\;\frac{E_\infty - E_A(T)}{E_A(T)}\right]$$

Table 2 is a continuation of Table 1 and gives the calculation of the coordinates of the interception contour by means of (2.7) and (2.5).

3. A Maximality Criterion

The following theorem gives a set of sufficient conditions for X's position to be in the interception contour. Unfortunately it is not applicable to the available energy function $E_A(T) = E_\infty(1 - e^{-\sqrt{T/b}})$. However, it is applicable to other available energy functions and suggests another way of looking at the main problem.

α and V shall remain fixed throughout the discussion. Reconsider Figure 1. It then is readily seen that fixing any two of the three

$b = 2.5$ $\sigma = 3$

$c = 1$ $E_\infty = 10^4$

T	$\frac{e^{-\sqrt{T/b}}}{1-e^{-\sqrt{T/b}}}$	$\left\{\frac{\sqrt{T/b}\,e^{-\sqrt{T/b}}}{1-e^{-\sqrt{T/b}}}\right\}$	$2(\sigma-1)+\{\ \}$	$\frac{U}{2\sigma}$	$\frac{d}{dT}(UT)$	$UT\left[\frac{1}{V}\frac{d}{dT}(UT)\right]$	$\left[\frac{1}{V}\frac{d}{dT}(UT)\right]^2$	$1-[\ ]^2$	$\sqrt{1-[\ ]^2}$	VT	$x = VT-UT[\ ]$	$y = UT\sqrt{1-[\ ]^2}$
3	0.5024	0.5501	4.5501	2.173	9.860	38.57	0.9722	.0278	.1667	30.	-8.57	6.521
4	3931	.4973	4.4973	2.025	9.107	44.26	.8294	.1706	.4130	40.	-4.26	20.07
5	.3214	4544	4.4544	1.913	8.521	48.91	.7261	.2939	.5134	50.	+1.09	29.47
6	.2698	.4179	4.4179	1.825	8.063	52.97	.6501	.3499	.5815	60.	7.03	38.20
7	.2311	.3866	4.3866	1.752	7.685	56.54	.5906	.4094	.6398	70	13.46	47.07
8	.2006	.3589	4.3589	1.688	7.358	59.63	.5414	.4586	.6772	80.	20.37	54.88
9	.1765	.3348	4.3348	1.635	7.087	62.58	.5022	.4978	.7055	90.	27.42	62.30
10	.1565	.3130	4.3130	1.588	6.849	65.25	.4691	.5309	.7286	100.	31.45	69.41
11	.1399	.2935	4.2935	1.546	6.638	67.71	.4406	.5594	.7479	110.	42.29	76.28
12	.1253	.2745	4.2745	1.508	6.446	70.00	.4155	.5845	.7645	120.	50.00	83.02
13	.1140	.2599	4.2599	1.473	6.275	72.10	.3938	.6062	.7786	130.	57.90	89.46
14	.1036	.2451	4.2451	1.442	6.121	74.12	.3747	.6253	.7908	140.	65.88	95.76
15	.0946	.232	4.232	1.413	5.980	76.06	.3576	.6424	.8015	150.	73.94	102.0
16	.0866	.219	4.219	1.386	5.848	77.78	.3420	.6580	.8112	160.	82.22	107.9
17	.0799	.208	4.208	1.361	5.727	79.49	.3280	.6720	.8198	170.	90.51	113.8
18	.0733	.197	4.197	1.338	5.616	81.15	.3154	.6846	.8274	180.	98.85	119.6
19	.0677	.187	4.187	1.316	5.510	82.70	.3036	.6964	.8345	190.	107.30	125.2
20	.0628	.178	4.178	1.295	5.410	84.07	.2927	.7073	.8410	200.	115.93	130.1
21	.0583	.169	4.169	1.277	5.324	85.66	.2845	.7166	.8465	210	124.34	136.2
22	.0543	.161	4.161	1.259	5.239	87.07	.2745	.7255	.8518	220.	132.93	141.6
23	.0506	.153	4.153	1.242	5.158	88.41	.2660	.7340	.8567	230.	141.59	146.8
24	.0472	.146	4.146	1.226	5.083	89.71	.2584	.7416	.8612	240.	150.29	152.0
25	.0442	.140	4.140	1.210	5.009	90.96	.2509	.7491	.8655	250.	159.04	157.2
26	.0414	.134	4.134	1.196	4.944	92.20	.2444	.7556	.8692	260.	167.80	162.1
27	.0388	.127	4.127	1.182	4.878	93.36	.2379	.7621	.8730	270.	176.64	167.1
28	.0365	.122	4.122	1.168	4.814	94.50	.2317	.7683	.8765	280.	185.50	172.0
29	.0343	.117	4.117	1.156	4.759	95.66	.2265	.7735	.8795	290.	194.34	176.8
30	.0323	.112	4.112	1.143	4.700	96.73	.2209	.7791	.8827	300.	203.27	181.6

TABLE 2.

variables D, β, and T uniquely determines the third. With this in mind we shall use the symbol $E_A(\beta, D)$ to mean the energy available to X when attempting an interception of Y when the situation is described by Figure 1. Similarly the symbol $E(\beta, D)$ shall mean the energy that X would expend, if he makes an interception of Y under the same circumstances.

Theorem: Let the function describing the available energy of X have the property that $E_A(T)/T$ is a strictly monotone decreasing function of T. If further there exists a $\tilde{\beta}$ and a $\tilde{D}$ such that $\dfrac{E_A(\tilde{\beta}, \tilde{D})}{E(\tilde{\beta}, \tilde{D})} = 1$ and $\dfrac{E_A(\beta, D)}{E(\beta, D)}$ attains its maximum at $\tilde{\beta}$ independently of D, then $\tilde{D}$ is maximal.

Proof: We consider any $D > \tilde{D}$ and show that $\dfrac{E_A(\beta, D)}{E(\beta, D)} < 1$ for all X i.e., it is impossible for X to intercept Y for $D > \tilde{D}$.

By the last assumption we have:

$$\frac{E_A(\beta, D)}{E(\beta, D)} \leq \frac{E_A(\tilde{\beta}, D)}{E(\tilde{\beta}, D)} \tag{3.1}$$

It is seen from Figure 7 that U' = U. Thus

$$\frac{E(\tilde{\beta}, \tilde{D})}{T} = \frac{cTU^{\sigma}}{T} = \frac{c(T+\Delta T)U'^{\sigma}}{T+\Delta T} = \frac{E(\tilde{\beta}, D)}{T+\Delta T} \tag{3.2}$$

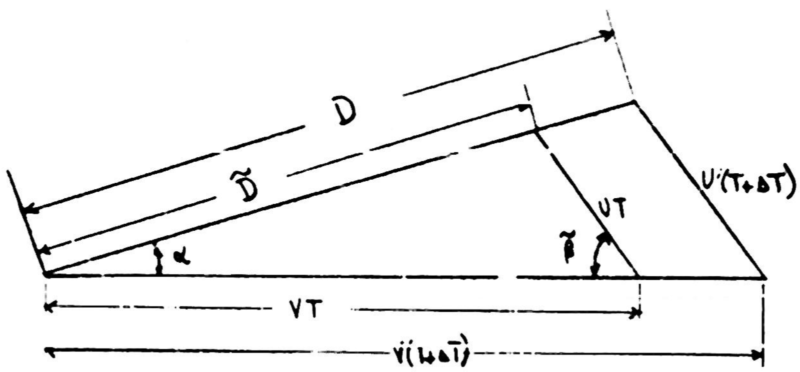


FIGURE 7

By the fi[illegible] [illegible]is [illegible] [illegible] [illegible]

$$(\ .\ ) \qquad \frac{E_A(\tilde{\beta}, D)}{T + \Delta T} < \frac{E_A(\tilde{\beta}, \tilde{D})}{T}$$

Upon [illegible]bining (.1), (.2), and (.3) [illegible]:

$$\frac{E_A(\beta, D)}{E(\beta, D)} \leq \frac{E_A(\tilde{\beta}, D)}{E(\tilde{\beta}, D)} < \frac{E_A(\tilde{\beta}, \tilde{D})}{E(\tilde{\beta}, \tilde{D})} = 1$$

which proves the theorem.

Let us consider the following possible available energy functions for X

a) $E_A(T) = K$

b) $E_A(T) = E_{\infty}\sqrt{T/b}$

c) $E_A(T) = E_{\infty}(1 - e^{-\sqrt{T/b}})$

d) $E_A(T) = KT$

It is fairly easy to check that $E_A(T)/T$ is a strictly monotone decreasing function for a), b), and c) and constant for d). Perhaps it should be remarked that this condition is equivalent to demanding that the maximum speed that X can travel for the time $T + \Delta T$ is less than the maximum speed that he is able to travel for the time T.

Suppose X's available energy function is $E_A(T) = K$. An application of the law of sines to Figure 1 gives:

$$(3.4) \qquad \frac{\sin(\alpha + \beta)}{D} = \frac{\sin \beta}{VT} = \frac{\sin \alpha}{UT}$$

Thus:

$$(3.5)\qquad \frac{E_A(\beta, D)}{E(\beta, D)} = \frac{K}{cTU^{\sigma}} = \frac{K \sin(\alpha + \beta)}{c D V^{\sigma - 1}} \frac{(\sin \beta)^{\sigma - 1}}{(\sin \alpha)^{\sigma}}$$

If we differentiate (3.5) with respect to β, we find that the result may be written in the form

$$(3.6)\qquad \frac{d}{d\beta} \frac{E_A(\beta, D)}{E(\beta, D)} = \frac{K \sin^{\sigma - 2} \beta}{c\, D V^{\sigma - 1} \sin^{\sigma} \alpha} \left[\sigma \sin(\alpha + \beta) \cos \beta - \sin \alpha \right]$$

Upon inspecting (3.6) we see that $\frac{E_A(\beta, D)}{E(\beta, D)}$ has minimums at $\beta = 0$ and $\beta = \pi$ and attains its maximum at $\overline{\beta}$ where

$$(3.7)\qquad \sigma \sin(\alpha + \overline{\beta}) \cos \overline{\beta} - \sin \alpha = 0.$$

If we assume $E_A(T) = E_{\infty} \sqrt{T/b}$, i.e., the first term in the expansion of $E_{\infty}(1 - e^{-\sqrt{T/b}})$, then upon proceding as in the last paragraph, it is easy to show that

$$(3.8)\qquad \frac{E_A(\beta, D)}{E(\beta, D)} = \frac{E_{\infty} \sqrt{\sin(\alpha + \beta)}\, \sin^{\sigma - \frac{1}{2}} \beta}{c \sqrt{b} \sqrt{D} \sin^{\sigma} \alpha \quad V^{\sigma - \frac{1}{2}}}$$

Again proceding as before, it is easy to see that the maximum of (3.8) occurs at β^* where

$$(3.9)\qquad 2\sigma \sin(\alpha + \beta^*) \cos \beta^* - \sin \alpha = 0.$$

If we attempt to find the maximum of $\frac{E_A(\beta, D)}{E(\beta, D)}$ with respect to β in the case where $E_A(T) = E_\infty(1 - e^{-\gamma T/b})$, we find that the maximum depends on D as well as on β.

If $E_A(T) = KT$, then

$$\frac{E_A(\beta, D)}{E(\beta, D)} = \frac{KT}{cTU^\sigma} - \frac{K}{c}\left(\frac{\sin\beta}{V \sin\alpha}\right)^\sigma$$

and this function has its maximum at $\beta = \pi/2$.

4. The Minimal Energy Contour.

In the last section we found the maximum of $K/E(\beta, D)$ to be $\overline{\beta}$ i.e., the angle which satisfies equation (3.7). Clearly $\overline{\beta}$ is also the minimum of $E(\beta, D)$ and thus it is the course X must employ if he is to use the least energy in making an interception. Hence it is of interest to determine exactly when it is possible for X to intercept Y if the interception course is $\overline{\beta}$.

We give the answer to this problem in terms of a second curve which we call the minimal energy contour. It enjoys the property that if X is inside of or on this contour, then he can intercept Y by using the course $\overline{\beta}$ and in so doing use a minimal amount of energy. If X lies outside of this contour, he still may be able to intercept Y, but he must adopt an interception course greater than $\overline{\beta}$ in order that he have enough energy available to make the interception.

We now determine the equation of this contour. From Figure 1 we see that the coordinates of this position may be found by finding the upper intersection point of the circles:

$$(4.1) \quad \begin{cases} x^2 + y^2 = D^2 \\ (x - VT)^2 + y^2 = (UT)^2 \end{cases}$$

It is

$$(4.2) \quad \begin{cases} x = \dfrac{(VT)^2 + D^2 - (UT)^2}{2\,VT} \\ y = \sqrt{(UT)^2 - (x - VT)^2} \end{cases}$$

Now equation (3.7) may be put in the form

$$(4.3) \qquad \cos \bar{\beta} = \frac{\sin \alpha}{\sigma \sin(\alpha + \bar{\beta})} = \frac{UT}{\sigma D}$$

An application of the law of cosines to Figure 1 gives

$$(4.4) \qquad (VT)^2 = D^2 + (UT)^2 - 2DUT \cos \bar{\beta}$$

Using (4.3) to eliminate $\cos \bar{\beta}$ from (4.4) and solving for D^2 , we obtain.

$$(4.5) \qquad D^2 = (VT)^2 - \left(\frac{\sigma - 2}{\sigma}\right) (UT)^2 .$$

Upon substituting (4.5) into (4.2) we find the equations of the minimal energy contour to be

$$(4.6) \quad \begin{cases} x = VT - \dfrac{\bar{U}T}{V} \dfrac{\sigma - 1}{\sigma} \bar{U} \\ y = \pm \bar{U}T \sqrt{1 - \left[\dfrac{\bar{U}}{V} \dfrac{\sigma - 1}{\sigma}\right]^2} \end{cases}$$

where $\bar{U}$ is given by the formula (2.1).

It should be noted that $\frac{\sigma - 1}{\sigma}\,\overline{U}$ is the first term of $\frac{d}{dT}\left[\overline{UT}\right]$ as given by (2.7) and that the minimal energy contour approaches the interception contour with increasing T.

We have seen that if X lies inside of or on the minimal energy contour, it is to his advantage to use the course $\overline{\beta}$ where

$$(3.7) \qquad \sigma \sin(\alpha + \overline{\beta}) \cos \overline{\beta} - \sin \alpha = 0.$$

Suppose we let $\overline{\beta} = \frac{1}{2}\left[\pi - \alpha + \gamma\right]$ and then substitute this expression into (3.7). Upon simplifying we obtain

$$\frac{\sigma}{2}(\sin \alpha - \sin \gamma) - \sin \alpha = 0.$$

Thus

$$(4.7) \qquad \sin \gamma = \frac{\sigma - 2}{\sigma} \sin \alpha$$

and

$$(4.8) \qquad \overline{\beta} = \frac{1}{2}\left[\pi - \alpha + \arcsin\left[\frac{\sigma - 2}{\sigma} \sin \alpha\right]\right]$$

The fact that $\overline{\beta}$ depends only upon α allows us to calculate the minimal energy contour as a function of α in a relatively simple manner.

Given α we determine $\overline{\beta}$ by (4.8). U is then given by $U = \frac{V \sin \alpha}{\sin \beta}$. To obtain the greatest D we must have the available energy equal to the energy used or equivalently that $\overline{U} = U$. The time T that gives us the $\overline{U}$ which satisfies this condition may be found by plotting $\overline{U}$ against time and finding the intersection of the resulting curve with the horizontal line $U = \frac{V \sin\alpha}{\sin\beta}$. Or since $\overline{U}T$ will probably have been plotted for the interception contour, the same ends may be accomplished by finding the intersection of the line $UT = \frac{V \sin\alpha}{\sin\beta} T$ with this curve. Once T is found D is given by $D = VT \frac{\sin(\alpha + \beta)}{\sin\beta}$.

5. Approximations to the Interception Contour

The minimal energy contour may also be thought of as an approximation to the interception contour. The reason for this is that the course $\overline{\beta}$ is the one X should use to maximize D, if his available energy is constant. Thus the minimal energy contour will lie close to the interception contour whenever the interception contour is generated by the nearly flat part of $E_\infty(1 - e^{-\sqrt{T/b}})$. If we let T_π be the interception time when X makes an angle of 180° with Y's bow, it is easily seen that T_π gives the first point of the interception contour in the sense that all other points are found by using times greater than T_π. Since T_π increases as V decreases, we see that the minimal energy contour will nearly coincide with the interception contour when V is small.

In Section 3 we saw that β^* is the course X should use to maximize D, if his available energy is given by $E_\infty\sqrt{T/b}$. Upon preceding as in Section 4, we find that the contour determined by the course β^*

and $\overline{UT}$ has the equations,

$$(5.1)\qquad \begin{cases} x = VT - \dfrac{UT}{V}\left(\dfrac{2\sigma - 1}{2\sigma}\right)\overline{U} \\ y = \pm\, \overline{U}T\sqrt{1 - \left\{\dfrac{2\sigma - 1}{2\sigma}\dfrac{\overline{U}}{V}\right\}^{2}} \end{cases}$$

and that β^* is given by the formula

$$(5.2)\qquad \beta^* = \tfrac{1}{2}\left\{\pi - \alpha + \arcsin\left[\frac{\sigma - 1}{\sigma}\sin\alpha\right]\right\}$$

For the lack of a better name we call this curve the parabolic energy contour. It is readily seen that this contour approximates the interception contour whenever the interception contour is determined by the nearly parabolic aprt of $E_\infty\,(1 - e^{-\sqrt{T/b}})$.

When X's position is on the interception contour, the course $\tilde{\beta}$ that he must use to intercept Y is determined by the normal to the interception contour at this point. It is fairly easy to establish from the geometry of the situation that

$$(5.3)\qquad \overline{\beta} \leq \tilde{\beta} \leq \beta^*$$

and that all three interception courses agree for $\alpha = 0$ and $\alpha = \pi$.

Figure 8 gives a graphical comparison of the minimal energy contour and the parabolic energy contour with the corresponding interception contour. The calculations for these curves were made by using the results of Tables 1 and 2.

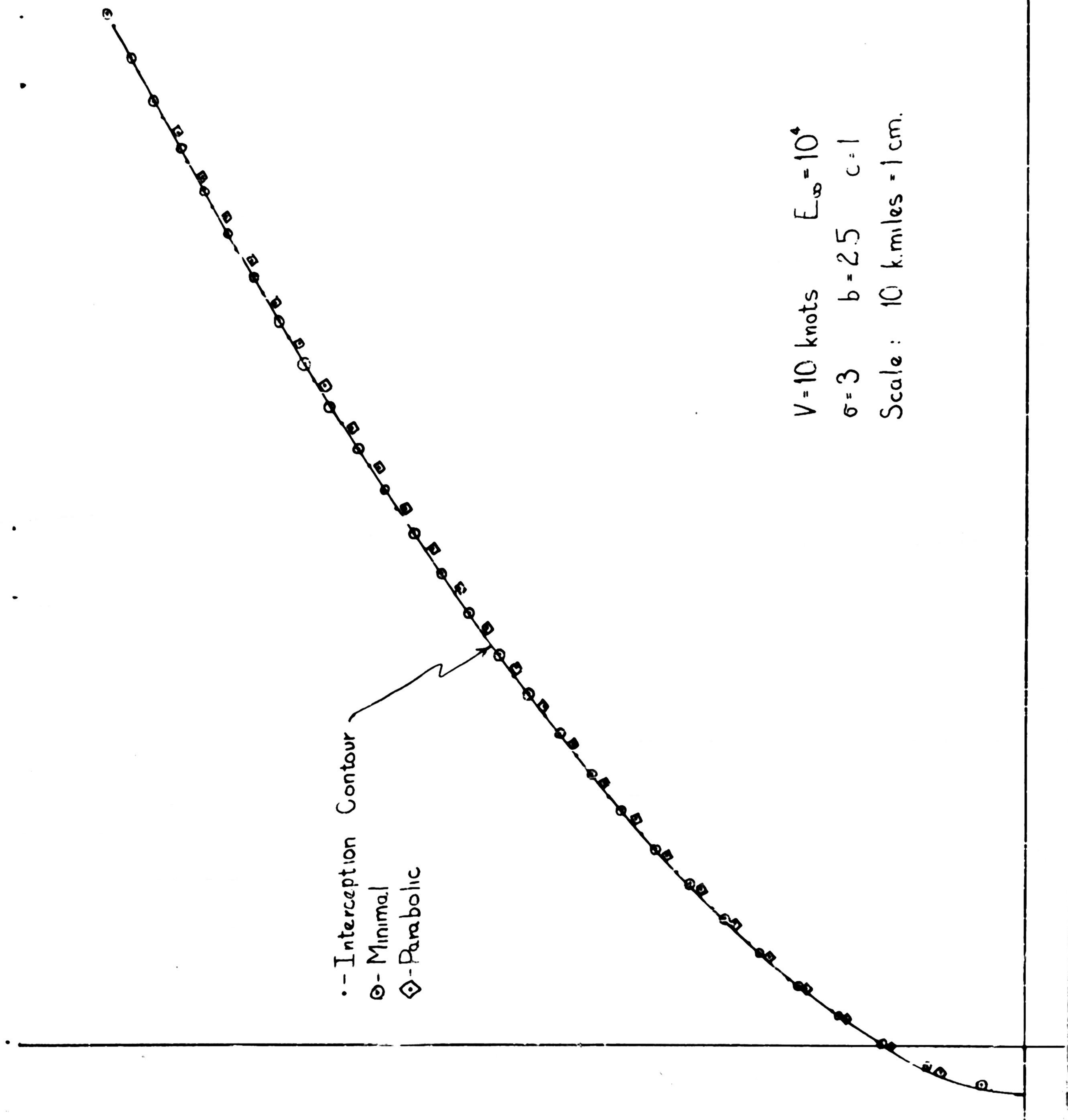

V = 10 knots $E_\infty = 10^4$
σ = 3   b = 2.5   c = 1
Scale: 10 k.miles = 1 cm.
· - Interception Contour
⊙ - Minimal
◇ - Parabolic

FIG 8

If $E_A(T) = KT$, as would be the case if X could obtains an unlimited amount of energy from his power supply, we find that $\bar{U} = (\frac{K}{c})^{1/\sigma}$, a constant. Substituting this value of $\bar{U}$ in (2.5), we see that the equation for X's interception contour becomes

$$(5.4) \qquad x = VT - \frac{\bar{U}^2 T}{V}$$
$$y = \pm \bar{U}T \sqrt{1 - (\frac{\bar{U}}{V})^2}$$

Upon eliminating T they become

$$(5.5) \qquad y = \pm \frac{\bar{U}}{\sqrt{V^2 - \bar{U}^2}} x$$

where $x \geq 0$ and $\bar{U} = (\frac{K}{c})^{1/\sigma}$. Thus if $\bar{U} < V$ the interception contour consists of two rays as illustrated by Figure 9. For $\bar{U} > V$, $(x < 0)$, the interception domain is the entire plane.

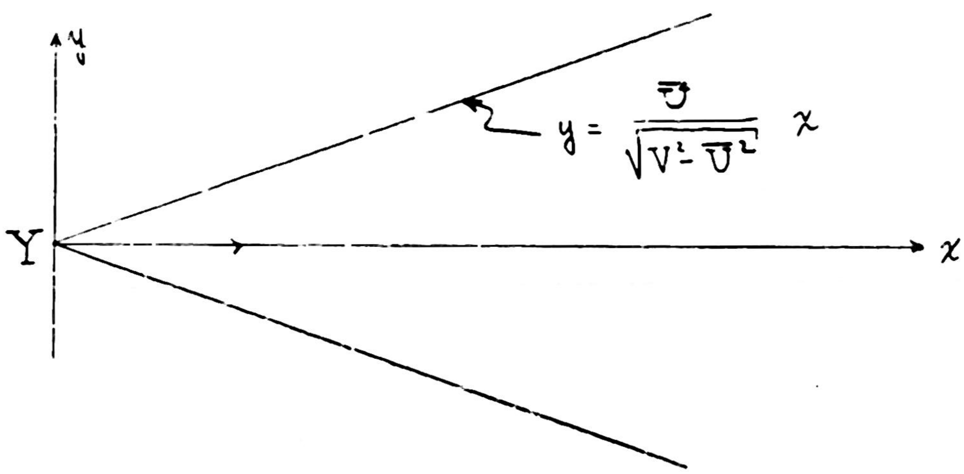


FIGURE 9

One of the principle reasons for inserting this section in the report is the thought that the function $E_\infty(1 - e^{-\sqrt{T/b}})$ might not always be an accurate description of X's available energy. If this should occur, then perhaps the above results can be of value.

6. The Interception Probability

The interception contour may be constructed about X as well as Y where, of course, it is assumed that Y has always the same course and speed regardless of position. This may be accomplished by placing X at the origin of the old interception contour and reversing Y's direction. In this section the interception contour will be taken relative to X.

Let us suppose that X can detect Y, if Y is within the range R of X. If we further assume X to be fixed for an indefinite period of time and Y to be uniformly distributed, then the probability of X successfully interception Y is equal to y'/R if $x' > 0$ and equal to 1 if $x' \leq 0$. Figure 10 illustrates the first situation.

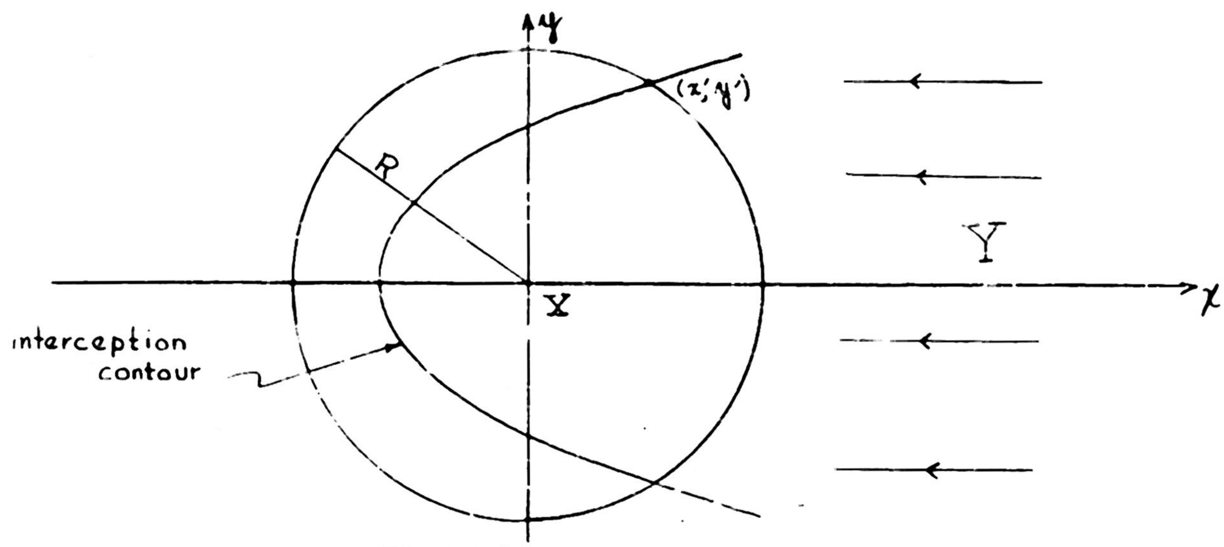


FIGURE 10

Thus, the interception probability (P_I) is readily calculated, once the point (x', y') is known. However, this point is the upper intersection point of $x^2 + y^2 = R^2$ with the interception contour, i.e., the point satisfying

$$(6.1)\quad \begin{cases} x^2 + y^2 = R^2 \\ x = VT - \frac{\overline{U}T}{V}\frac{d}{dT}\left[\overline{U}T\right] \\ y = \overline{U}T\sqrt{1 - \left\{\frac{1}{V}\frac{d}{dT}\left[\overline{U}T\right]\right\}^2} \end{cases}$$

This point may be found by eliminating x and y in (6.1) and finding the T that satisfies the resulting expression in V, $\overline{U}$ and $\frac{d}{dT}\left[\overline{U}T\right]$.

This T may then be inserted in the last equation of (6.1) to find y'. It appears to us that this procedure involves as much labor as the construction of the interception contour from which the probability may readily be determined.

However, if the minimal energy contour is used in place of the interception contour, then the above method may be employed with considerable success. In this case we wish to find the (x',y') which satisfies

$$(6.2)\quad \begin{cases} x^2 + y^2 = R^2 \\ x = VT - \frac{\overline{U}T}{V}\left(\frac{\sigma - 1}{\sigma}\right)\overline{U} \\ y = \overline{U}T\sqrt{1 - \left\{\frac{\overline{U}}{V}\frac{\sigma - 1}{\sigma}\right\}^2} \end{cases}$$

Eliminating x and y in (6.2), we obtain

$$(6.3)\quad (VT)^2 - \frac{\sigma - 2}{\sigma}(\overline{U}T)^2 = R^2$$

or

$$(6.4)\quad (\overline{U}T) = V\sqrt{\frac{\sigma}{\sigma - 2}}\sqrt{T^2 - \left(\frac{R}{V}\right)^2}$$

The right hand side of (6.4) is a hyperbola and thus graphical means may be used to determine the desired T. Figure 11 illustrates this method.

RESTRICTED

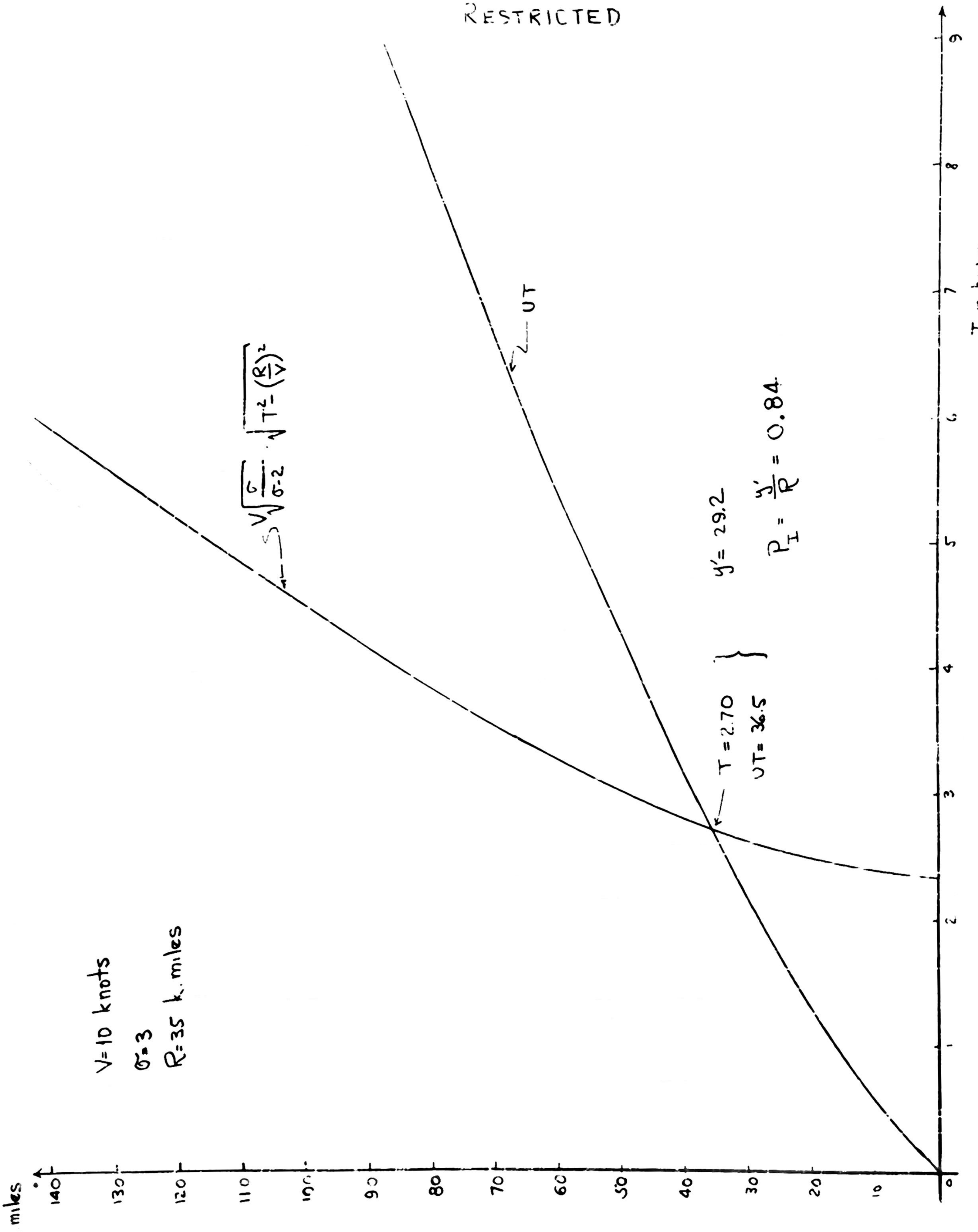


FIG. 11

APPENDIX

Following is a list of the partial derivatives of the interception contour and the minimal energy contour with respect to the parameters σ, E_∞, b, and c.

(1) $\dfrac{\partial}{\partial \sigma}$:

Interception contour.

$$\begin{cases} \dfrac{\partial x}{\partial \sigma} = \dfrac{2TU\eta}{\sigma} \log U - \dfrac{U^2 T}{V\sigma^2}\left(1 - \dfrac{\frac{1}{2}\sqrt{T/b}}{e^{\sqrt{T/b}} - 1}\right) \\ \dfrac{\partial y}{\partial \sigma} = \dfrac{1}{\sqrt{1-\eta^2}}\left(\dfrac{UT}{\sigma} \log U + \eta \dfrac{\partial x}{\partial \sigma}\right) \text{ where } \eta = \dfrac{1}{V}\dfrac{d}{dT}(UT) \end{cases}$$

Minimal energy contour

$$\begin{cases} \dfrac{\partial x}{\partial \sigma} = \dfrac{U^2 T}{V\sigma^2}\left[2(\sigma - 1)\log U - 1\right] \\ \dfrac{\partial y}{\partial \sigma} = \pm \dfrac{UT}{\sigma\sqrt{1-\varsigma^2}}\left[(1 - 2\varsigma^2)\log U - \dfrac{U}{\sigma}\right] \text{ where } \varsigma = \dfrac{U}{V}\dfrac{\sigma - 1}{\sigma} \end{cases}$$

(2) $\dfrac{\partial}{\partial E_\infty}$:

Interception contour

$$\begin{cases} \dfrac{\partial x}{\partial E_\infty} = -\dfrac{2\eta}{\sigma c} U^{\frac{1-\sigma}{\sigma}}\left(1 - e^{-\sqrt{T/b}}\right) \\ \dfrac{\partial y}{\partial E_\infty} = \pm \dfrac{1 - 2\eta^2}{2\eta\sqrt{1-\eta^2}} \dfrac{\partial x}{\partial E_\infty} \end{cases}$$

Minimal energy contour

$$\begin{cases} \dfrac{\partial x}{\partial E_\infty} = 0 \\ \dfrac{\partial y}{\partial E_\infty} = 0 \end{cases}$$

(3) $\dfrac{\partial}{\partial b}$:

Interception contour

$$\dfrac{\partial x}{\partial b} = -2T\eta\dfrac{\partial U}{\partial b} - \dfrac{U^2 T}{V\sigma}\left[\dfrac{1}{4\sqrt{Tb}\left(e^{\sqrt{T/b}} - 1\right)} - \dfrac{e^{\sqrt{T/b}}}{4b\left(e^{\sqrt{T/b}} - 1\right)^2}\right]$$

$$\frac{\partial y}{\partial b} = \pm \frac{1}{\sqrt{1-\eta^2}} \left(\eta \frac{\partial x}{\partial b} + T \frac{\partial U}{\partial b} \right) \text{ where } \frac{\partial U}{\partial b} = \frac{E_\infty e^{-\gamma}}{2\sigma\tau T \pi h}$$

Minimal energy contour

$$\frac{\partial x}{\partial b} = 0$$

$$\frac{\partial y}{\partial b} = 0$$

(4) $\frac{\partial}{\partial c}$:

Interception contour

$$\frac{\partial x}{\partial c} = -\frac{1}{c} \frac{\partial x}{\partial E_\infty}$$

$$\frac{\partial y}{\partial c} = -\frac{1}{c} \frac{\partial y}{\partial E_\infty}$$

Minimal energy contour

$$\frac{\partial x}{\partial c} = \frac{TU}{c\sigma} 2\varsigma$$

$$\frac{\partial y}{\partial c} = \pm \frac{TU}{c\sigma} \frac{1-2\varsigma^2}{\sqrt{1-\varsigma^2}}$$